

PRIMARY AUTOMORPHIC FORMS: THETA CONSTANTS, THE j -INVARIANT, MODULAR LAMBDA, THE DYADIC ISOGENY CORRESPONDENCE, AND FOUR NEW STRUCTURAL OBSERVATIONS

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ABSTRACT. We develop the central identities of the theory of automorphic forms centering on the Jacobi theta constants $\vartheta_2, \vartheta_3, \vartheta_4$, the weight-4 Eisenstein series E_4 , the discriminant Δ , the j -invariant, and the modular λ -function. The classical theory is organized around a single minimality theorem: the pair $(\vartheta_3(\tau), \vartheta_3(2\tau))$ suffices to recover every primary automorphic invariant at level ≤ 2 as an explicit polynomial or rational function.

Building on this foundation, we derive four new structural observations. **(I)** The shifted invariant $J(\tau) := j(\tau) - 744$ satisfies $J(2\tau) = J(\tau)^2 - 2 \cdot 196884 + O(q^2)$ (and in fact $J(2\tau) = J(\tau)^2 - 2 \cdot 196884 - 2 \cdot 21493760 q^2 + O(q^4)$), placing the first Monster moonshine coefficient as the *leading deviation from perfect squaring* under the doubling isogeny; the corresponding quadratic fixed-point polynomial has discriminant 1575073. **(II)** The sequence $\vartheta_3(2^n \tau)^2$ is the arithmetic-mean sequence of the arithmetic-geometric mean (AGM) iteration initialized at $(\vartheta_3(\tau)^2, \vartheta_4(\tau)^2)$; the unique AGM fixed-point symmetry $\vartheta_3(\tau) = \vartheta_4(\tau)$ identifies $j(\tau) = 1728$ ($\tau = i$) as the self-dual elliptic curve. **(III)** The λ -ODE $d\lambda/dt = -\pi\lambda(1-\lambda)\vartheta_3(it)^4$ approaches a logistic regime for $t \gg 1$; matching the exact midpoint value $\lambda(i) = 1/2$ produces the explicit sigmoid approximation $\lambda(it) \approx (1 + e^{\pi(t-1)})^{-1}$ for large t . **(IV)** The quantity $R(\tau) := 2\vartheta_3(2\tau)^2 - \vartheta_3(\tau)^2 = \vartheta_4(\tau)^2$ satisfies the square-root recursion $R(2\tau) = \vartheta_3(\tau)\sqrt{R(\tau)}$ under doubling; equivalently, $\vartheta_4(2^n \tau)$ lies in a nested-radical (generically quadratic) extension tower over the dyadic ϑ_3 -field, growing by one quadratic layer at each step—an algebraic obstruction distinct from the polynomial j -isogeny ladder.

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1. INTRODUCTION

1.1. Scope and purpose. The theory of modular forms for $\mathrm{SL}_2(\mathbb{Z})$ is built on a small number of fundamental objects: the Eisenstein series, the discriminant cusp form, the Jacobi theta constants, and the automorphic invariants they generate. This paper assembles the primary identities relating these objects into a single self-contained account, then derives four new structural observations from first principles.

The guiding organizing principle is *polynomial expressibility*: which automorphic invariants can be expressed as explicit polynomials or rational functions of a minimal set of generating theta data? The classical answer is complete: the pair $(A, B) = (\vartheta_3(\tau), \vartheta_3(2\tau))$ determines λ , E_4 , Δ , and j via the duplication identities. Sections 2–7 establish this classical foundation.

Sections 8–11 then take this foundation seriously and push it, deriving four structural observations:

- (I) **Shifted squaring and moonshine:** $J := j - 744$ nearly squares under $\tau \mapsto 2\tau$, with correction term 2×196884 (Section 8).
- (II) **AGM structure of the theta sequence:** the dyadic theta orbit is the arithmetic-mean sequence of a classical AGM (Section 9).
- (III) **Universal sigmoid crossover of λ :** the λ -ODE has a logistic regime for $t \gg 1$; matching at $t = 1$ yields an explicit sigmoid approximation (Section 10).
- (IV) **Square-root recursion and the quadratic extension tower:** the quantity $R(\tau) = \vartheta_4(\tau)^2$ satisfies a square-root recursion under doubling, generating a nested-radical tower for $\vartheta_4(2^n \tau)$ (Section 11).

1.2. Organization. Sections 2–7 develop the classical theory efficiently but completely, with full proofs where they illuminate structure. Each of Sections 8–11 is self-contained: it states an observation, gives a complete derivation from the classical material, and discusses consequences and further directions.

2. THE UPPER HALF-PLANE AND THE MODULAR GROUP

Definition 2.1 (Upper half-plane and nome). The *upper half-plane* is $\mathbb{H} := \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$. For $\tau \in \mathbb{H}$ the *nome* is $q := e^{\pi i \tau}$; since $\text{Im}(\tau) > 0$ we have $|q| < 1$.

Definition 2.2 (Modular group and action). The *modular group* $\text{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by $\gamma \cdot \tau = (a\tau + b)/(c\tau + d)$. Its generators are $T : \tau \mapsto \tau + 1$ and $S : \tau \mapsto -1/\tau$. The effective group is $\text{PSL}_2(\mathbb{Z}) := \text{SL}_2(\mathbb{Z})/\{\pm I\}$.

Definition 2.3 (Modular forms and automorphic functions). A holomorphic $f : \mathbb{H} \rightarrow \mathbb{C}$ is a *modular form of weight k* if $f(\gamma \cdot \tau) = (c\tau + d)^k f(\tau)$ for all $\gamma \in \text{SL}_2(\mathbb{Z})$, and f is holomorphic at the cusp. Weight- $k = 0$ invariants are *automorphic functions*.

3. JACOBI THETA CONSTANTS

Definition 3.1 (Theta constants). For $\tau \in \mathbb{H}$ and $q = e^{\pi i \tau}$:

$$\vartheta_2(\tau) := 2 \sum_{n \geq 0} q^{(n + \frac{1}{2})^2}, \quad (3.1)$$

$$\vartheta_3(\tau) := 1 + 2 \sum_{n \geq 1} q^{n^2}, \quad (3.2)$$

$$\vartheta_4(\tau) := 1 + 2 \sum_{n \geq 1} (-1)^n q^{n^2}. \quad (3.3)$$

Theorem 3.2 (Jacobi identity). $\vartheta_3(\tau)^4 = \vartheta_2(\tau)^4 + \vartheta_4(\tau)^4$.

Theorem 3.3 (Transformation formulas under $S : \tau \mapsto -1/\tau$).

$$\vartheta_2(-1/\tau) = (-i\tau)^{1/2} \vartheta_4(\tau), \quad (3.4)$$

$$\vartheta_3(-1/\tau) = (-i\tau)^{1/2} \vartheta_3(\tau), \quad (3.5)$$

$$\vartheta_4(-1/\tau) = (-i\tau)^{1/2} \vartheta_2(\tau). \quad (3.6)$$

Theorem 3.4 (q -product representations).

$$\vartheta_2(\tau) = 2q^{1/4} \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n})^2, \quad (3.7)$$

$$\vartheta_3(\tau) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^{2n-1})^2, \quad (3.8)$$

$$\vartheta_4(\tau) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - q^{2n-1})^2. \quad (3.9)$$

4. EISENSTEIN SERIES, DISCRIMINANT, AND j -INVARIANT

Definition 4.1 (Primary automorphic objects).

$$E_4(\tau) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n}, \quad (4.1)$$

$$\Delta(\tau) := q^2 \prod_{n=1}^{\infty} (1 - q^{2n})^{24}, \quad (4.2)$$

$$j(\tau) := \frac{E_4(\tau)^3}{\Delta(\tau)} = q^{-2} + 744 + 196884 q^2 + \dots \quad (4.3)$$

E_4 is a weight-4 modular form; Δ is the unique (up to scalar) weight-12 cusp form, nonvanishing on $\mathbb{H}$; j is the $\mathrm{PSL}_2(\mathbb{Z})$ -invariant automorphic function giving a biholomorphism $\mathrm{PSL}_2(\mathbb{Z}) \backslash (\mathbb{H} \cup \{\infty\}) \xrightarrow{\sim} \mathbb{P}^1(\mathbb{C})$.

Theorem 4.2 (Theta representations of E_4 , Δ , j).

$$E_4(\tau) = \frac{1}{2} (\vartheta_2(\tau)^8 + \vartheta_3(\tau)^8 + \vartheta_4(\tau)^8), \quad (4.4)$$

$$\Delta(\tau) = \frac{1}{256} \vartheta_2(\tau)^8 \vartheta_3(\tau)^8 \vartheta_4(\tau)^8, \quad (4.5)$$

$$j(\tau) = 32 \frac{(\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8)^3}{\vartheta_2^8 \vartheta_3^8 \vartheta_4^8}. \quad (4.6)$$

5. THE MODULAR λ -FUNCTION AND THE j - λ RELATION

Definition 5.1 (Modular λ -function).

$$\lambda(\tau) := \left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)} \right)^4. \quad (5.1)$$

This is a Hauptmodul for $\Gamma(2)$, giving a biholomorphism $\Gamma(2) \backslash \mathbb{H} \xrightarrow{\sim} \mathbb{C} \setminus \{0, 1\}$. On the imaginary axis, $\lambda(it) \in (0, 1)$ for all $t > 0$.

Proposition 5.2 (Functional equation and special values of λ). *The modular λ -function satisfies*

$$\lambda\left(-\frac{1}{\tau}\right) = 1 - \lambda(\tau). \quad (5.2)$$

In particular, $\lambda(i) = 1/2$. Moreover, on the imaginary axis:

$$\lambda(it) \rightarrow 1 \quad (t \rightarrow 0^+), \quad \lambda(it) \rightarrow 0 \quad (t \rightarrow +\infty). \quad (5.3)$$

Proof. From Theorem 3.3,

$$\lambda\left(-\frac{1}{\tau}\right) = \left(\frac{\vartheta_2(-1/\tau)}{\vartheta_3(-1/\tau)} \right)^4 = \left(\frac{\vartheta_4(\tau)}{\vartheta_3(\tau)} \right)^4.$$

Using Jacobi's identity $\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4$ gives

$$1 - \lambda(\tau) = 1 - \left(\frac{\vartheta_2}{\vartheta_3}\right)^4 = \left(\frac{\vartheta_4}{\vartheta_3}\right)^4,$$

hence (5.2). Setting $\tau = i$ yields $\lambda(i) = 1 - \lambda(i)$ so $\lambda(i) = 1/2$. For $t \rightarrow +\infty$, $q = e^{-\pi t} \rightarrow 0$ so $\vartheta_2(it) \rightarrow 0$ and $\vartheta_3(it) \rightarrow 1$, hence $\lambda(it) \rightarrow 0$. For $t \rightarrow 0^+$, use (5.2) with $\tau = it$: $\lambda(it) = 1 - \lambda(i/t)$ and $i/t \rightarrow i\infty$ so $\lambda(i/t) \rightarrow 0$. $\square$

Theorem 5.3 (j as a rational function of λ ; polynomial relation).

$$j(\tau) = 256 \frac{(1 - \lambda + \lambda^2)^3}{\lambda^2(1 - \lambda)^2}. \quad (5.4)$$

Equivalently, j and λ satisfy the irreducible degree-6 polynomial relation

$$j \lambda^2 (1 - \lambda)^2 - 256 (1 - \lambda + \lambda^2)^3 = 0. \quad (5.5)$$

Proposition 5.4 (Differential equation for λ). For $t > 0$ with $\tau = it$,

$$\frac{d\lambda}{dt} = -\pi \lambda (1 - \lambda) \vartheta_3(it)^4. \quad (5.6)$$

6. THETA DUPLICATION IDENTITIES AND TWO-SCALE FORMULAS

Theorem 6.1 (Theta duplication identities).

$$\vartheta_3(\tau)^2 = \vartheta_3(2\tau)^2 + \vartheta_2(2\tau)^2, \quad (6.1)$$

$$\vartheta_4(\tau)^2 = \vartheta_3(2\tau)^2 - \vartheta_2(2\tau)^2, \quad (6.2)$$

$$\vartheta_2(\tau)^2 = 2 \vartheta_2(2\tau) \vartheta_3(2\tau). \quad (6.3)$$

Theorem 6.2 (Two-scale minimality: (A, B) determines all primary invariants). Let $A := \vartheta_3(\tau)$ and $B := \vartheta_3(2\tau)$. Then:

$$\vartheta_2(\tau)^8 = 16 B^4 (A^2 - B^2)^2, \quad (6.4)$$

$$\vartheta_4(\tau)^8 = (A^2 - 2B^2)^4, \quad (6.5)$$

and

$$\lambda(\tau) = \frac{4B^2(A^2 - B^2)}{A^4}, \quad (6.6)$$

$$1 - \lambda(\tau) = \frac{(A^2 - 2B^2)^2}{A^4}, \quad (6.7)$$

$$E_4(\tau) = \frac{1}{2} (A^8 + (A^2 - 2B^2)^4 + 16B^4(A^2 - B^2)^2), \quad (6.8)$$

$$\Delta(\tau) = \frac{1}{16} B^4 (A^2 - B^2)^2 A^8 (A^2 - 2B^2)^4, \quad (6.9)$$

so that $j(\tau) = E_4(\tau)^3 / \Delta(\tau)$ is a quotient of polynomials in (A, B) .

Proof. From (6.1), $\vartheta_2(2\tau)^2 = A^2 - B^2$. From (6.3), $\vartheta_2(\tau)^4 = 4B^2(A^2 - B^2)$; squaring gives (6.4). From (6.2), $\vartheta_4(\tau)^2 = B^2 - (A^2 - B^2) = 2B^2 - A^2$, so $\vartheta_4(\tau)^4 = (A^2 - 2B^2)^2$; squaring gives (6.5). The formulas for λ and $1 - \lambda$ follow by dividing by $A^4 = \vartheta_3(\tau)^4$. Substituting into Theorem 4.2 gives (6.8)–(6.9). $\square$

7. DYADIC ISOGENY AND THE MODULAR POLYNOMIAL Φ_2

Definition 7.1 (Modular polynomial Φ_2).

$$\begin{aligned}\Phi_2(X, Y) := & X^3 + Y^3 - X^2Y^2 + 1488(X^2Y + XY^2) \\ & - 162000(X^2 + Y^2) + 40773375XY \\ & + 8748000000(X + Y) - 157464000000000.\end{aligned}\tag{7.1}$$

Theorem 7.2 (2-isogeny algebraic identity). *For all $\tau \in \mathbb{H}$, $\Phi_2(j(\tau), j(2\tau)) = 0$. The polynomial Φ_2 is symmetric, of degree 3 in each variable, corresponding to the three cyclic order-2 subgroups of $(\mathbb{Z}/2\mathbb{Z})^2$.*

Definition 7.3 (Dyadic orbit). Fix $\tau_0 \in \mathbb{H}$. Set $\tau_n := 2^n\tau_0$. Consecutive j -values satisfy $\Phi_2(j(\tau_n), j(\tau_{n+1})) = 0$ for all $n \geq 0$, forming an *algebraic ladder*.

8. OBSERVATION I: SHIFTED SQUARING AND MONSTER MOONSHINE

8.1. Statement.

Observation 8.1 (Shifted squaring identity). *Define the shifted j -invariant*

$$J(\tau) := j(\tau) - 744.\tag{8.1}$$

Then, with $q = e^{\pi i \tau}$,

$$\boxed{J(2\tau) = J(\tau)^2 - 2 \cdot 196884 + O(q^2)}.\tag{8.2}$$

More precisely,

$$\boxed{J(2\tau) = J(\tau)^2 - 2 \cdot 196884 - 2 \cdot 21493760 q^2 + O(q^4)}.\tag{8.3}$$

The leading correction coefficient 2×196884 is twice the dimension of the smallest non-trivial representation of the Monster group $\mathbb{M}$.

8.2. Derivation. From the standard q -expansion (with $q = e^{\pi i \tau}$, so j has terms $q^{-2}, q^0, q^2, \dots$):

$$j(\tau) = q^{-2} + 744 + 196884 q^2 + 21493760 q^4 + 864299970 q^6 + \dots\tag{8.4}$$

Hence

$$J(\tau) = q^{-2} + 196884 q^2 + 21493760 q^4 + 864299970 q^6 + \dots.\tag{8.5}$$

Under $\tau \mapsto 2\tau$, the nome $q \mapsto q^2$:

$$J(2\tau) = q^{-4} + 196884 q^4 + 21493760 q^8 + O(q^{12}).\tag{8.6}$$

On the other hand, squaring gives

$$\begin{aligned}J(\tau)^2 &= (q^{-2} + 196884 q^2 + 21493760 q^4 + 864299970 q^6 + \dots)^2 \\ &= q^{-4} + 2 \cdot 196884 + 2 \cdot 21493760 q^2 + (196884^2 + 2 \cdot 864299970) q^4 + O(q^6).\end{aligned}\tag{8.7}$$

Subtracting,

$$J(2\tau) - J(\tau)^2 = -2 \cdot 196884 - 2 \cdot 21493760 q^2 + (196884^2 - 2 \cdot 864299970) q^4 + O(q^6),\tag{8.8}$$

which implies (8.2) and (8.3).

8.3. The fixed-point polynomial and its discriminant.

Proposition 8.2 (Fixed points and discriminant). *The leading-order quadratic approximation to the isogeny doubling map on J is*

$$f(x) := x^2 - 2 \cdot 196884 = x^2 - 393768. \quad (8.9)$$

The fixed points of $f(x) = x$ are the roots of

$$x^2 - x - 393768 = 0, \quad (8.10)$$

which has discriminant

$$\Delta_{\text{fp}} = 1 + 4 \cdot 393768 = 1575073. \quad (8.11)$$

In particular, the fixed points are real but (since Δ_{fp} is not a square) are not rational.

Remark 8.3. Including the q^2 correction from (8.3), one has

$$J(2\tau) = J(\tau)^2 - 393768 - 42987520 q^2 + O(q^4). \quad (8.12)$$

The coefficient $42987520 = 2 \cdot 21493760$ is the next moonshine coefficient in the q -expansion of J .

Remark 8.4 (Moonshine interpretation). In Monstrous Moonshine (Conway–Norton, Borchers), the coefficients of $J(\tau)$ are dimensions of graded pieces of the Monster module $V^\natural$: $\dim V_n^\natural = c(n)$ where $J(\tau) = \sum c(n)q^{2n}$. Observation 8.1 states that the discrepancy between $J(2\tau)$ and $J(\tau)^2$ is, to leading order, $2c(1) = 2 \dim V_1^\natural$, i.e., twice the dimension of the first nontrivial graded piece. This suggests an interpretation: the failure of J to behave multiplicatively with respect to the doubling correspondence is measured in moonshine units.

A natural question: for the McKay–Thompson series $T_g(\tau)$ of an element $g \in \mathbb{M}$, does $T_g(2\tau) - T_g(\tau)^2 = -2\chi_g + O(q)$ where χ_g is the character of g on $V_1^\natural$? This would be a moonshine-level refinement of (8.2).

9. OBSERVATION II: THE AGM STRUCTURE OF THE DYADIC THETA SEQUENCE

9.1. Statement.

Observation 9.1 (Dyadic theta orbit as AGM arithmetic-mean sequence). *Fix $\tau \in \mathbb{H}$ and let*

$$a_n := \vartheta_3(2^n \tau)^2, \quad d_n := \vartheta_4(2^n \tau)^2.$$

Then the duplication identities imply the exact AGM recursion

$$\boxed{a_{n+1} = \frac{a_n + d_n}{2}, \quad d_{n+1} = \sqrt{a_n d_n}.} \quad (9.1)$$

(For $\tau = it$ on the imaginary axis, the square root is taken as the positive real root.) In particular, a_n is the arithmetic-mean sequence of the classical AGM iteration initialized at $(a_0, d_0) = (\vartheta_3(\tau)^2, \vartheta_4(\tau)^2)$, and (a_n, d_n) converges rapidly to the common limit 1.

9.2. Derivation. From the duplication identity (6.1)–(6.2) one obtains

$$\vartheta_3(2\tau)^2 = \frac{\vartheta_3(\tau)^2 + \vartheta_4(\tau)^2}{2}, \quad (9.2)$$

$$\vartheta_4(2\tau)^2 = \vartheta_3(\tau)\vartheta_4(\tau), \quad (9.3)$$

where (9.3) is the geometric-mean identity (proved from q -products in the Appendix). Identity (9.2) says: $\vartheta_3(2\tau)^2$ is the arithmetic mean of $\vartheta_3(\tau)^2$ and $\vartheta_4(\tau)^2$. Identity (9.3) says: $\vartheta_4(2\tau)^2$ is the geometric mean of $\vartheta_3(\tau)^2$ and $\vartheta_4(\tau)^2$. Thus the pair (a_n, d_n) iterates the AGM step exactly, giving (9.1).

Proposition 9.2 (AGM limit and convergence rate). *For any $\tau = it$ with $t > 0$:*

- (i) $\text{AGM}(\vartheta_3(it)^2, \vartheta_4(it)^2) = 1$.
- (ii) As $n \rightarrow \infty$,

$$\vartheta_3(2^n it)^2 - 1 \sim 4e^{-\pi \cdot 2^n t}, \quad 1 - \vartheta_4(2^n it)^2 \sim 4e^{-\pi \cdot 2^n t}. \quad (9.4)$$

Proof. (i) Since $2^n it \rightarrow i\infty$, we have $q_n = e^{-\pi 2^n t} \rightarrow 0$ and hence $\vartheta_3(2^n it) \rightarrow 1$ and $\vartheta_4(2^n it) \rightarrow 1$. The AGM recursion forces both sequences to converge to their common AGM limit, which is therefore 1.

(ii) For large n , $\vartheta_3(2^n it) = 1 + 2q_n + O(q_n^4)$, so $\vartheta_3(2^n it)^2 = 1 + 4q_n + O(q_n^2)$ and $\vartheta_3(2^n it)^2 - 1 \sim 4q_n$. Similarly $\vartheta_4(2^n it) = 1 - 2q_n + O(q_n^4)$ gives $\vartheta_4(2^n it)^2 = 1 - 4q_n + O(q_n^2)$ and $1 - \vartheta_4(2^n it)^2 \sim 4q_n$. $\square$

9.3. The self-dual fixed point at $\tau = i$.

Proposition 9.3 (Self-dual fixed point). *The AGM iteration (a_n, d_n) initialized at $(\vartheta_3(\tau)^2, \vartheta_4(\tau)^2)$ is symmetric ($a_0 = d_0$) if and only if $\vartheta_3(\tau) = \vartheta_4(\tau)$, which holds if and only if $\lambda(\tau) = 1/2$, equivalently $j(\tau) = 1728$, equivalently $\tau = i$ (in the fundamental domain).*

Proof. $a_0 = d_0 \Leftrightarrow \vartheta_3(\tau)^2 = \vartheta_4(\tau)^2$. On the imaginary axis both ϑ_3, ϑ_4 are positive, so $a_0 = d_0 \Leftrightarrow \vartheta_3 = \vartheta_4$. From $\lambda = (\vartheta_2/\vartheta_3)^4$ and $\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4$:

$$\vartheta_3 = \vartheta_4 \Rightarrow \vartheta_3^4 = \vartheta_4^4 \Rightarrow \vartheta_2^4 = \vartheta_4^4 \Rightarrow \lambda = 1/2.$$

Then (5.4) gives $j = 1728$. Finally, $j(\tau) = 1728$ occurs at $\tau = i$ in the standard fundamental domain. $\square$

Remark 9.4 (Geometric meaning). $j = 1728$ corresponds to the elliptic curve $y^2 = x^3 - x$, which has the extra automorphism $[i] : (x, y) \mapsto (-x, iy)$ of order 4. This is precisely the unique elliptic curve (up to isomorphism) that is isomorphic to its own quadratic twist, i.e., the *self-dual* curve. Proposition 9.3 says this special geometric property is encoded as the unique symmetric starting point of the theta AGM.

10. OBSERVATION III: UNIVERSAL SIGMOID CROSSOVER OF λ

10.1. Statement.

Observation 10.1 (Sigmoid approximation from the large- t logistic regime). *Consider the ODE (5.6): $d\lambda/dt = -\pi\lambda(1-\lambda)\vartheta_3(it)^4$ for $\lambda = \lambda(it)$, $t > 0$.*

- (i) For $t \rightarrow 0^+$: $\vartheta_3(it)^4 \sim t^{-2}$ and $\lambda(it) \rightarrow 1$.
- (ii) For $t \rightarrow \infty$: $\vartheta_3(it)^4 \rightarrow 1$, and the ODE approaches the logistic equation $d\lambda/dt = -\pi\lambda(1-\lambda)$.
- (iii) Matching the exact midpoint value $\lambda(i) = 1/2$ to the logistic-limit family yields the explicit sigmoid

$$\lambda(it) \approx \frac{1}{1 + e^{\pi(t-1)}}, \quad (10.1)$$

which is a convenient large- t closed-form approximation (with center at $t = 1$).

10.2. Derivation. Step 1: Behavior at $t \rightarrow 0^+$. From the S -transform $\vartheta_3(-1/\tau) = (-i\tau)^{1/2}\vartheta_3(\tau)$, setting $\tau = it$:

$$\vartheta_3(i/t) = t^{1/2}\vartheta_3(it). \quad (10.2)$$

As $t \rightarrow 0^+$, $i/t \rightarrow i\infty$, so $\vartheta_3(i/t) \rightarrow 1$. Therefore $\vartheta_3(it) \sim t^{-1/2}$ and $\vartheta_3(it)^4 \sim t^{-2}$. By Proposition 5.2, $\lambda(it) \rightarrow 1$ as $t \rightarrow 0^+$.

Step 2: Behavior at $t \rightarrow \infty$. $q = e^{-\pi t} \rightarrow 0$, so $\vartheta_3(it) = 1 + 2e^{-\pi t} + \dots \rightarrow 1$ and $\vartheta_3(it)^4 \rightarrow 1$. The ODE approaches

$$\frac{d\lambda}{dt} \approx -\pi\lambda(1-\lambda), \quad (10.3)$$

the standard logistic equation, whose general solution is $\lambda(t) = 1/(1 + Ce^{\pi t})$ for constant $C > 0$.

Step 3: Fixing the constant by $\lambda(i) = 1/2$. We know $\lambda(i) = 1/2$ exactly (Proposition 5.2). Substituting $t = 1$, $\lambda = 1/2$ into $1/(1 + Ce^\pi) = 1/2$ gives $C = e^{-\pi}$. Therefore

$$\lambda(it) \approx \frac{1}{1 + e^{-\pi} e^{\pi t}} = \frac{1}{1 + e^{\pi(t-1)}}, \quad (10.4)$$

which is (10.1).

Proposition 10.2 (Quantitative control via the logit variable). *Define the logit variable $y(t) := \log(\lambda(it)/(1 - \lambda(it)))$. Then*

$$\frac{dy}{dt} = -\pi \vartheta_3(it)^4. \quad (10.5)$$

Hence for any $T \geq 1$ and all $t \geq T$,

$$y(t) = y(T) - \pi(t - T) - \pi \int_T^t (\vartheta_3(is)^4 - 1) ds. \quad (10.6)$$

Moreover, for $t \geq 1$ one has the elementary bound

$$0 < \vartheta_3(it)^4 - 1 \leq 16e^{-\pi t}, \quad (10.7)$$

so the deviation of $y(t)$ from its logistic-limit behavior is bounded by

$$0 \leq \pi \int_T^t (\vartheta_3(is)^4 - 1) ds \leq 16e^{-\pi T}. \quad (10.8)$$

Remark 10.3 (Crossover and the three special values). The point $t = 1$ simultaneously realizes: (i) the unique fixed point of $t \mapsto 1/t$ (the imaginary-axis analogue of S); (ii) the self-dual elliptic curve $j = 1728$; (iii) the exact symmetry value $\lambda(i) = 1/2$; (iv) the center of the logistic-limit sigmoid (10.1) obtained by matching at $t = 1$. These coincidences have a common cause: the S -symmetry $\tau \mapsto -1/\tau$ fixes $\tau = i$ and forces $\lambda(i) = 1 - \lambda(i)$, i.e., $\lambda(i) = 1/2$.

11. OBSERVATION IV: SQUARE-ROOT RECURSION AND THE QUADRATIC EXTENSION TOWER

11.1. Statement.

Observation 11.1 (Square-root recursion of $R = \vartheta_4^2$). *Define*

$$R(\tau) := 2\vartheta_3(2\tau)^2 - \vartheta_3(\tau)^2 = \vartheta_4(\tau)^2. \quad (11.1)$$

Then

$$\boxed{R(2\tau) = \vartheta_3(\tau) \sqrt{R(\tau)}}. \quad (11.2)$$

Consequently, the n -fold iterate satisfies

$$\boxed{R(2^n \tau) = \left(\prod_{k=0}^{n-1} \vartheta_3(2^k \tau)^{1/2^{n-1-k}} \right) \cdot R(\tau)^{1/2^n}}, \quad (11.3)$$

a nested-radical expression. In particular, while each square $R(2^n \tau) = \vartheta_4(2^n \tau)^2$ is polynomially expressible in the dyadic ϑ_3 -data, the unsquared values $\vartheta_4(2^n \tau) = \sqrt{R(2^n \tau)}$ generically generate a tower of quadratic extensions growing by one layer at each step—a genuinely different algebraic structure from the polynomial j -isogeny ladder.

11.2. Derivation. First, using (6.1)–(6.2),

$$\vartheta_3(2\tau)^2 = \frac{\vartheta_3(\tau)^2 + \vartheta_4(\tau)^2}{2} \implies 2\vartheta_3(2\tau)^2 - \vartheta_3(\tau)^2 = \vartheta_4(\tau)^2,$$

which is (11.1).

Next, the geometric-mean identity (9.3) gives

$$\vartheta_4(2\tau)^2 = \vartheta_3(\tau)\vartheta_4(\tau).$$

In terms of $R(\tau) = \vartheta_4(\tau)^2$, this is

$$R(2\tau) = \vartheta_4(2\tau)^2 = \vartheta_3(\tau)\vartheta_4(\tau) = \vartheta_3(\tau)\sqrt{R(\tau)},$$

establishing (11.2) (with an understood choice of square-root branch; on the imaginary axis, take the positive real root).

Proposition 11.2 (Nested-radical formula for the full orbit). *Iterating (11.2) yields*

$$R(2\tau) = \vartheta_3(\tau) \cdot R(\tau)^{1/2}, \tag{11.4}$$

$$R(4\tau) = \vartheta_3(2\tau) \cdot \vartheta_3(\tau)^{1/2} \cdot R(\tau)^{1/4}, \tag{11.5}$$

$$R(8\tau) = \vartheta_3(4\tau) \cdot \vartheta_3(2\tau)^{1/2} \cdot \vartheta_3(\tau)^{1/4} \cdot R(\tau)^{1/8}, \tag{11.6}$$

and in general (11.3).

Proof. By induction: set $R_n := R(2^n\tau)$ and $A_n := \vartheta_3(2^n\tau)$. Then $R_{n+1} = A_n\sqrt{R_n}$. Expanding recursively gives the stated pattern of exponents, and the closed form (11.3) follows. $\square$

Theorem 11.3 (A quadratic extension tower for $\vartheta_4(2^n\tau)$ over dyadic ϑ_3). *Fix $\tau \in \mathbb{H}$ and let*

$$F_0 = \mathbb{C}(\vartheta_3(2^n\tau) : n \geq 0).$$

For $n \geq 0$ define

$$F_{n+1} := F_n(\vartheta_4(2^n\tau)) = F_n(\sqrt{R(2^n\tau)}).$$

Then:

- (i) *For each n , $[F_{n+1} : F_n] \leq 2$.*
- (ii) *For each n , $R(2^n\tau) = \vartheta_4(2^n\tau)^2 \in F_0$ (indeed $R(2^n\tau) = 2\vartheta_3(2^{n+1}\tau)^2 - \vartheta_3(2^n\tau)^2$).*
- (iii) *Generically (i.e. away from special algebraic loci such as CM points and accidental square relations), each adjunction $\vartheta_4(2^n\tau)$ produces a genuine quadratic extension, so the tower does not stabilize after finitely many steps.*

Remark 11.4 (Contrast with the j -ladder). The j -isogeny ladder $\{j(2^n\tau)\}$ is governed by Φ_2 , a polynomial relation of degree 3. Each $j(2^n\tau)$ is algebraic of degree $\leq 3^n$ over $\mathbb{C}(j(\tau))$. By contrast, the tower in Theorem 11.3 is radical in nature: even though $\vartheta_4(2^n\tau)^2$ is polynomially expressible in dyadic ϑ_3 -data, recovering $\vartheta_4(2^n\tau)$ itself introduces (generically) a new square root at each step. The j -ladder is polynomial-algebraic; the ϑ_4 -tower is radical-algebraic. These are genuinely different algebraic structures coexisting inside the same automorphic orbit.

12. SUMMARY AND FURTHER DIRECTIONS

12.1. Summary table. The following table collects the primary automorphic objects and their polynomial sources, together with the new observations.

Object	Source / formula	Section
$\vartheta_3(\tau), \vartheta_3(2\tau)$	Primary data (A, B)	§3
$\lambda(\tau)$	$4B^2(A^2 - B^2)/A^4$	§6
$E_4(\tau)$	Polynomial in (A, B) , eq. (6.8)	§6
$\Delta(\tau)$	Polynomial in (A, B) , eq. (6.9)	§6
$j(\tau)$	Rational in (A, B) ; $j\lambda^2(1 - \lambda)^2 = 256(1 - \lambda + \lambda^2)^3$	§5
$J(2\tau) - J(\tau)^2$	$-2 \cdot 196884 + O(q^2)$ (moonshine correction)	§8
$(a_n, d_n) = (\vartheta_3(2^n\tau)^2, \vartheta_4(2^n\tau)^2)$	AGM iteration; limit = 1	§9
$\lambda(it)$	$\approx 1/(1 + e^{\pi(t-1)})$ for $t \gg 1$	§10
$\vartheta_4(2^n\tau)$	Nested radical / quadratic tower over dyadic ϑ_3	§11

12.2. Open questions. Each observation opens a concrete research direction.

From Observation I. For an element g of the Monster group $\mathbb{M}$, let $T_g(\tau) = q^{-1} + \sum c_g(n)q^n$ be the McKay–Thompson series. Does $T_g(2\tau) - T_g(\tau)^2 = -2c_g(1) + O(q)$? More generally, what is the full correction series, and does it encode the character table of $\mathbb{M}$ on $V^\natural$?

From Observation II. The AGM of $(\vartheta_3^2, \vartheta_4^2)$ equals 1. The *geometric* AGM limit is $\prod_{n=0}^{\infty} \vartheta_4(2^n\tau)/\vartheta_3(2^n\tau)$, which relates to the complete elliptic integral $K(\lambda^{1/2})$ via $K = (\pi/2)\vartheta_3(\tau)^2$. The AGM convergence rate gives a quantitative bound on the theta-spectrum decay. What is the spectrum of convergence exponents across $\tau \in \mathbb{H}$?

From Observation III. The sigmoid approximation (10.1) gives a completely explicit approximate formula for $\lambda(it)$ on the imaginary axis. How well does the logistic model approximate $\lambda(it)$ for moderate t ? The deviation from pure logistic behavior is controlled explicitly by $\vartheta_3(it)^4 - 1$ (Proposition 10.2). Can one leverage such closed forms to build simple approximations to modular quantities built from λ ?

From Observation IV. The quadratic extension tower generated by $\{\vartheta_4(2^n\tau)\}$ over the dyadic ϑ_3 -field is an explicit, concrete radical tower attached to automorphic data. What is its Galois group (as a profinite limit) for generic τ ? Does it embed into a known Galois-theoretic object such as the absolute Galois group of $\mathbb{Q}$ via the Galois action on CM values of ϑ_4 ?

13. ADDENDUM TO §8: MCKAY–THOMPSON SERIES AND THE DOUBLING SQUARING DEFECT

This section addresses the open questions following Observation 8.1 for the McKay–Thompson series T_g of an element $g \in \mathbb{M}$.

13.1. Normalization conventions ($q = e^{\pi i\tau}$ **versus** $q = e^{2\pi i\tau}$). Throughout the main body of this paper we used

$$q := e^{\pi i\tau},$$

so $j(\tau) = q^{-2} + 744 + 196884q^2 + \dots$.

In Monstrous Moonshine it is standard to use

$$Q := e^{2\pi i\tau} = q^2,$$

so that

$$J(\tau) = j(\tau) - 744 = Q^{-1} + 196884Q + 21493760Q^2 + \dots$$

Similarly, a McKay–Thompson series is conventionally written

$$T_g(\tau) = Q^{-1} + \sum_{n \geq 1} c_g(n) Q^n, \tag{13.1}$$

with vanishing constant term (the genus-zero Hauptmodul normalization).

All statements below are most transparent in the Q -normalization; converting back to the paper's q -normalization is done by $Q = q^2$ and $T_g(\tau) = \sum a_m q^m$ with only odd/even powers as appropriate.

13.2. The universal leading term: $T_g(2\tau) - T_g(\tau)^2 = -2c_g(1) + O(Q)$.

Proposition 13.1 (Leading discrepancy from squaring under doubling). *Let T_g be as in (13.1). Then*

$$T_g(2\tau) - T_g(\tau)^2 = -2c_g(1) + O(Q). \quad (13.2)$$

In particular, the constant term of $T_g(2\tau) - T_g(\tau)^2$ is $-2c_g(1)$.

Proof. From (13.1),

$$T_g(2\tau) = Q^{-2} + \sum_{n \geq 1} c_g(n) Q^{2n}.$$

Also,

$$\begin{aligned} T_g(\tau)^2 &= \left(Q^{-1} + \sum_{n \geq 1} c_g(n) Q^n \right)^2 \\ &= Q^{-2} + 2c_g(1) + O(Q). \end{aligned}$$

Subtracting yields (13.2). □

Remark 13.2 (Representation-theoretic meaning). In Monstrous Moonshine one has

$$c_g(n) = \text{Tr}\left(g \mid V_n^{\natural}\right),$$

so the constant term in (13.2) is

$$-2c_g(1) = -2 \text{Tr}\left(g \mid V_1^{\natural}\right).$$

Thus the leading deviation from perfect squaring under the doubling map encodes the character of g on the first nontrivial graded piece $V_1^{\natural}$.

13.3. The full correction series: explicit coefficient formula. Define the *doubling squaring defect* of T_g by

$$D_g(\tau) := T_g(2\tau) - T_g(\tau)^2 = \sum_{m \geq 0} d_g(m) Q^m. \quad (13.3)$$

Proposition 13.3 (Explicit coefficients of D_g). *Let T_g be as in (13.1). Then for each $m \geq 0$,*

$$d_g(m) = \mathbf{1}_{2|m} c_g(m/2) - \left(2c_g(m+1) + \sum_{\substack{a+b=m \\ a, b \geq 1}} c_g(a)c_g(b) \right), \quad (13.4)$$

where $\mathbf{1}_{2|m}$ is 1 if m is even and 0 otherwise. In particular,

$$d_g(0) = -2c_g(1), \quad d_g(1) = -2c_g(2), \quad d_g(2) = c_g(1) - (2c_g(3) + c_g(1)^2).$$

Proof. Write $T_g(2\tau) = Q^{-2} + \sum_{n \geq 1} c_g(n) Q^{2n}$, so the coefficient of Q^m in $T_g(2\tau)$ is $\mathbf{1}_{2|m} c_g(m/2)$ for $m \geq 0$. Next expand

$$T_g(\tau)^2 = Q^{-2} + 2 \sum_{n \geq 1} c_g(n) Q^{n-1} + \sum_{a, b \geq 1} c_g(a)c_g(b) Q^{a+b}.$$

The coefficient of Q^m in the middle term is $2c_g(m+1)$ and in the double sum is $\sum_{a+b=m, a, b \geq 1} c_g(a)c_g(b)$. Subtracting gives (13.4). □

13.4. Does D_g encode the Monster character table?

Remark 13.4 (What D_g can and cannot add). The family of functions $\{T_g(\tau)\}$ (over all conjugacy classes of $\mathbb{M}$) already encodes the characters $\text{Tr}(g \mid V_n^{\frac{1}{2}})$ for every n by definition. Therefore D_g does not provide additional *linear* information beyond the $c_g(n)$, since each $d_g(m)$ is a universal polynomial expression in the $c_g(\cdot)$ via (13.4).

What D_g can encode are nonlinear constraints among traces. In the genus-zero setting, the deeper structure is Norton's *replicability*: doubling is controlled not only by $T_g(2\tau)$ and $T_g(\tau)^2$, but by Hecke-like combinations involving $\tau/2$ and $(\tau+1)/2$ and the power map $g \mapsto g^2$. From that viewpoint, D_g is a coarse but explicit “shadow” of replicability.

14. ADDENDUM TO §9: CONVERGENCE EXPONENTS FOR THE THETA-AGM ACROSS $\mathbb{H}$

This section makes precise the dependence of the dyadic theta-AGM convergence rate on τ .

14.1. Dyadic convergence is governed by the dyadically rescaled nome. Let $q := e^{\pi i \tau}$ as in the paper. Then $|q| = e^{-\pi \Im(\tau)}$. For the dyadic orbit $\tau_n := 2^n \tau$ we have the dyadic nome

$$q_n := e^{\pi i \tau_n} = q^{2^n}, \quad |q_n| = e^{-\pi 2^n \Im(\tau)}.$$

Proposition 14.1 (Uniform dyadic exponential control). *For any $\tau \in \mathbb{H}$ and all $n \geq 0$,*

$$\vartheta_3(2^n \tau) - 1 = 2q_n + O(q_n^4), \quad (14.1)$$

$$\vartheta_4(2^n \tau) - 1 = -2q_n + O(q_n^4). \quad (14.2)$$

Consequently,

$$\vartheta_3(2^n \tau)^2 - 1 = 4q_n + O(q_n^2), \quad (14.3)$$

$$1 - \vartheta_4(2^n \tau)^2 = 4q_n + O(q_n^2). \quad (14.4)$$

The implied constants are absolute (coming from the absolutely convergent theta series).

Proof. From Definition 3.1,

$$\vartheta_3(2^n \tau) = 1 + 2 \sum_{m \geq 1} q_n^{m^2} = 1 + 2q_n + 2q_n^4 + \cdots$$

and similarly

$$\vartheta_4(2^n \tau) = 1 + 2 \sum_{m \geq 1} (-1)^m q_n^{m^2} = 1 - 2q_n + 2q_n^4 - \cdots,$$

which gives (14.1)–(14.2). Squaring and using $|q_n| < 1$ yields (14.3)–(14.4). $\square$

14.2. A “spectrum” of convergence exponents. A convenient summary statistic for convergence is the asymptotic slope of $\log |\vartheta_3(2^n \tau)^2 - 1|$ as a function of 2^n .

Definition 14.2 (Dyadic convergence exponent). For $\tau \in \mathbb{H}$ define the dyadic convergence exponent

$$\alpha(\tau) := \pi \Im(\tau). \quad (14.5)$$

Proposition 14.3 (Asymptotic exponent law). *Fix $\tau \in \mathbb{H}$. Then as $n \rightarrow \infty$,*

$$\log |\vartheta_3(2^n \tau)^2 - 1| = -\alpha(\tau) 2^n + O(1), \quad (14.6)$$

and similarly

$$\log |1 - \vartheta_4(2^n \tau)^2| = -\alpha(\tau) 2^n + O(1). \quad (14.7)$$

Proof. By Proposition 14.1,

$$\vartheta_3(2^n \tau)^2 - 1 = 4q^{2^n} + O(|q|^{2 \cdot 2^n}),$$

so $|\vartheta_3(2^n \tau)^2 - 1| = 4|q|^{2^n} (1 + o(1))$. Taking logs gives (14.6) since $\log |q| = -\pi \Im(\tau) = -\alpha(\tau)$. $\square$

Remark 14.4 (Interpretation). Thus across $\tau \in \mathbb{H}$ the “spectrum” of exponents is the continuous family $\alpha(\tau) = \pi\Im(\tau)$: larger imaginary part gives exponentially faster dyadic convergence. Quadratic (AGM) convergence in the iteration index n manifests as exponential convergence in 2^n .

15. ADDENDUM TO §10: QUANTITATIVE LOGISTIC APPROXIMATION FOR $\lambda(it)$

This section refines the open questions following Observation 10.1 by providing explicit comparison inequalities and an asymptotic calibration discussion.

15.1. The logit transform yields an exact integral identity. Let $\lambda(t) := \lambda(it)$ for $t > 0$ and define the logit variable

$$y(t) := \log\left(\frac{\lambda(t)}{1 - \lambda(t)}\right). \quad (15.1)$$

Proposition 15.1 (Exact logit evolution). *For all $t > 0$,*

$$y'(t) = -\pi \vartheta_3(it)^4, \quad (15.2)$$

and with the exact midpoint $\lambda(1) = \lambda(i) = 1/2$ (Proposition 5.2), one has $y(1) = 0$ and

$$y(t) = -\pi \int_1^t \vartheta_3(is)^4 ds. \quad (15.3)$$

Proof. Differentiate (15.1):

$$y' = \frac{\lambda'}{\lambda} + \frac{\lambda'}{1 - \lambda} = \frac{\lambda'}{\lambda(1 - \lambda)}.$$

Insert $\lambda' = -\pi\lambda(1 - \lambda)\vartheta_3(it)^4$ from (5.6) to obtain (15.2), then integrate using $y(1) = 0$. $\square$

15.2. Logistic comparison: the sigmoid is an upper bound for $t \geq 1$. Define the matched logistic sigmoid

$$\lambda_{\log}(t) := \frac{1}{1 + e^{\pi(t-1)}}. \quad (15.4)$$

Proposition 15.2 (One-sided comparison for $t \geq 1$). *For all $t \geq 1$,*

$$\lambda(it) \leq \lambda_{\log}(t). \quad (15.5)$$

Equivalently, $y(t) \leq -\pi(t - 1)$.

Proof. For all $t > 0$ one has $\vartheta_3(it) > 1$, hence $\vartheta_3(it)^4 > 1$. Thus from (15.2), $y'(t) < -\pi$ for $t > 0$. With $y(1) = 0$, integration gives $y(t) \leq -\pi(t - 1)$ for $t \geq 1$. Exponentiating and converting back to λ gives (15.5). $\square$

15.3. Explicit error control from bounds on $\vartheta_3(it)^4 - 1$.

Lemma 15.3 (Elementary upper bound on $\vartheta_3(it)^4 - 1$ for $t \geq 1$). *For $t \geq 1$,*

$$0 < \vartheta_3(it)^4 - 1 \leq \frac{16e^{-\pi t}}{1 - e^{-\pi t}}. \quad (15.6)$$

In particular, $0 < \vartheta_3(it)^4 - 1 \leq 32e^{-\pi t}$ for all $t \geq 1$.

Proof. Let $x := e^{-\pi t} \in (0, e^{-\pi}]$. From the series $\vartheta_3(it) = 1 + 2\sum_{n \geq 1} x^{n^2}$ we bound

$$0 < \vartheta_3(it) - 1 \leq 2 \sum_{n \geq 1} x^n = \frac{2x}{1 - x}.$$

For $u \geq 0$, $(1 + u)^4 - 1 \leq 4u(1 + u)^3 \leq 4u(1 + u)^3$ and with $u = \vartheta_3 - 1 \leq 2x/(1 - x)$ one obtains a bound of the form (15.6); the stated constants are safe (not sharp) and suffice for explicit integral control. $\square$

Proposition 15.4 (Uniform logit error bound for $t \geq 1$). *For all $t \geq 1$,*

$$0 \leq (-\pi(t-1) - y(t)) = \pi \int_1^t (\vartheta_3(is)^4 - 1) ds \leq C_1 e^{-\pi}, \quad (15.7)$$

where one may take $C_1 = 32$ (non-sharp), and more generally for any $T \geq 1$ and all $t \geq T$,

$$0 \leq (-\pi(t-T) - (y(t) - y(T))) \leq C_1 e^{-\pi T}. \quad (15.8)$$

Proof. Combine (15.3) with Lemma 15.3 and integrate the geometric bound $e^{-\pi s}$ on $[T, \infty)$. $\square$

15.4. Asymptotic calibration: why the matched sigmoid is only a rough large- t model.

Proposition 15.5 (True large- t asymptotic of $\lambda(it)$). *As $t \rightarrow +\infty$,*

$$\lambda(it) = 16e^{-\pi t} - 128e^{-2\pi t} + O(e^{-3\pi t}). \quad (15.9)$$

Remark 15.6 (Comparison with the matched logistic sigmoid). From (15.4),

$$\lambda_{\log}(t) \sim e^{-\pi(t-1)} = e^{\pi} e^{-\pi t} \quad (t \rightarrow \infty).$$

Since the true leading constant in (15.9) is 16, the matched sigmoid has asymptotic constant $e^{\pi} \approx 23.14$, i.e. a relative large- t error approaching $e^{\pi}/16 - 1 \approx 0.446$. Thus (10.1) should be viewed as a convenient *closed-form surrogate* matched at $t = 1$, not as an asymptotically sharp model. One can obtain an asymptotically calibrated logistic by shifting the center to $t_0 := \pi^{-1} \log 16 \approx 0.882$, giving $\lambda(t) \approx (1 + e^{\pi(t-t_0)})^{-1}$ for large t , though this no longer matches $\lambda(i) = 1/2$ exactly.

Remark 15.7 (Using closed forms to approximate modular quantities built from λ). Any modular quantity that is a rational function of λ (e.g. $j(\tau)$ via Theorem 5.3) can be approximated on the imaginary axis by substituting an explicit surrogate for $\lambda(it)$, such as (15.4) or an asymptotically calibrated variant. Proposition 15.1 makes the approximation problem equivalent to approximating the integral of $\vartheta_3(it)^4$, for which explicit exponentially small tails are available.

16. ADDENDUM TO §11: THE RADICAL TOWER AND ITS (PRO-2) GALOIS STRUCTURE

This section addresses the open questions following Observation 11.1: the expected Galois structure for generic τ and the CM (singular value) regime.

16.1. Reformulation as a Kummer tower over the dyadic ϑ_3 -field. Fix $\tau \in \mathbb{H}$ and define the base field

$$F_0 := \mathbb{C}(\vartheta_3(2^n \tau) : n \geq 0). \quad (16.1)$$

Define the radicands

$$r_n := R(2^n \tau) = \vartheta_4(2^n \tau)^2 = 2\vartheta_3(2^{n+1} \tau)^2 - \vartheta_3(2^n \tau)^2 \in F_0. \quad (16.2)$$

Now define the tower

$$F_{n+1} := F_n(\vartheta_4(2^n \tau)) = F_n(\sqrt{r_n}). \quad (16.3)$$

Proposition 16.1 (Kummer nature and finite-stage Galois groups). *For each $N \geq 1$, the compositum field*

$$F_0(\sqrt{r_0}, \sqrt{r_1}, \dots, \sqrt{r_{N-1}})$$

is a Kummer extension of exponent 2 over F_0 . In particular, its Galois closure over F_0 has Galois group a finite 2-group, and if the square classes of $r_0, \dots, r_{N-1}$ are independent in $F_0^{\times}/(F_0^{\times})^2$, then the Galois group is the elementary abelian group

$$(\mathbb{Z}/2\mathbb{Z})^N$$

acting by independent sign changes on the $\sqrt{r_n}$.

Proof. Over any field of characteristic $\neq 2$, adjoining finitely many square roots produces a Kummer extension of exponent 2. If the radicands are independent modulo squares, then the automorphisms are exactly independent sign changes, giving $(\mathbb{Z}/2\mathbb{Z})^N$. $\square$

16.2. The profinite limit for “generic” τ .

Definition 16.2 (Pro-2 Galois group of the radical tower (generic heuristic)). Assuming the radicands $\{r_n\}_{n \geq 0}$ are independent modulo squares in $F_0^\times$, the inverse system of finite-stage Galois groups yields a profinite group

$$G_\infty \cong \prod_{n \geq 0} \mathbb{Z}/2\mathbb{Z}, \quad (16.4)$$

the profinite product of countably many $\mathbb{Z}/2\mathbb{Z}$.

Remark 16.3 (What “generic” should mean). A fully rigorous statement requires specifying a class of τ for which no accidental square relations $r_n \in (F_0(\sqrt{r_0}, \dots, \sqrt{r_{n-1}})^\times)^2$ occur. One expects such relations only on thin loci (special algebraic constraints, e.g. CM points). Proving independence is a Kummer-theoretic problem over the transcendental field F_0 .

16.3. CM points and embedding into class field theory. Now assume τ is CM, i.e. τ is imaginary quadratic and $K := \mathbb{Q}(\tau)$ is an imaginary quadratic field.

Remark 16.4 (CM singular values and Shimura reciprocity). Values of modular functions at CM points are algebraic and generate explicit class fields. For $\Gamma(2)$ -level invariants one has:

- $\lambda(\tau)$ is algebraic and lies in a suitable ring/ray class field of K .
- Theta constants (or closely related eta-quotients and Weber functions) at CM points produce classical class invariants; their Galois action is described by Shimura reciprocity.

Therefore, when τ is CM, the algebraic fields generated by $\vartheta_4(2^n \tau)$ (after appropriate normalization to remove transcendental scaling) are expected to lie in a tower of ring/ray class fields of K of 2-power conductor. In this regime the relevant Galois groups are abelian over K , controlled by (ray) class groups, and hence embed into $G_\mathbb{Q}$ through the canonical restriction map $G_\mathbb{Q} \rightarrow \text{Gal}(H/K)$ for the appropriate class field H .

Remark 16.5 (Answer to the embedding question). Thus:

- For generic (non-CM) τ , the natural expectation is a large pro-2 Kummer group of the form (16.4) (subject to independence).
- For CM τ , the tower is governed by explicit CM class field theory and embeds into $G_\mathbb{Q}$ in the standard way via Shimura reciprocity; the resulting extensions over K are expected to be (largely) abelian, reflecting the CM nature rather than producing a new nonabelian piece of $G_\mathbb{Q}$.

APPENDIX A. EXPLICIT COEFFICIENT TABLE FOR $J(2\tau) - J(\tau)^2$

Writing $q = e^{\pi i \tau}$:

$$\begin{aligned} J(\tau) &= q^{-2} + 196884q^2 + 21493760q^4 + 864299970q^6 + \dots \\ J(\tau)^2 &= q^{-4} + 2 \cdot 196884 + 2 \cdot 21493760q^2 + (196884^2 + 2 \cdot 864299970)q^4 + \dots \\ J(2\tau) &= q^{-4} + 196884q^4 + 21493760q^8 + \dots \\ J(2\tau) - J(\tau)^2 &= -393768 - 42987520q^2 + (196884 - 196884^2 - 2 \cdot 864299970)q^4 + \dots \end{aligned}$$

The constant term is $-393768 = -2 \times 196884$. The coefficient of q^2 is $-42987520 = -2 \times 21493760$. The coefficient of q^4 is

$$196884 - 196884^2 - 2 \cdot 864299970 = -40491712512.$$

APPENDIX B. THE GEOMETRIC-MEAN IDENTITY $\vartheta_4(2\tau)^2 = \vartheta_3(\tau)\vartheta_4(\tau)$

We give a direct proof from q -products (Theorem 3.4). Let $q = e^{\pi i \tau}$.

From Theorem 3.4,

$$\begin{aligned}\vartheta_3(\tau)\vartheta_4(\tau) &= \prod_{n=1}^{\infty} (1 - q^{2n})^2 (1 + q^{2n-1})^2 (1 - q^{2n-1})^2 \\ &= \prod_{n=1}^{\infty} (1 - q^{2n})^2 (1 - q^{4n-2})^2.\end{aligned}$$

Also,

$$\vartheta_4(2\tau) = \prod_{n=1}^{\infty} (1 - q^{4n})(1 - q^{4n-2})^2,$$

so

$$\vartheta_4(2\tau)^2 = \prod_{n=1}^{\infty} (1 - q^{4n})^2 (1 - q^{4n-2})^4.$$

Finally, split the product for $\vartheta_3(\tau)\vartheta_4(\tau)$ into even and odd indices:

$$\prod_{n=1}^{\infty} (1 - q^{2n})^2 = \left(\prod_{m=1}^{\infty} (1 - q^{4m})^2 \right) \left(\prod_{m=1}^{\infty} (1 - q^{4m-2})^2 \right).$$

Therefore

$$\begin{aligned}\vartheta_3(\tau)\vartheta_4(\tau) &= \left(\prod_{m=1}^{\infty} (1 - q^{4m})^2 \right) \left(\prod_{m=1}^{\infty} (1 - q^{4m-2})^2 \right) \left(\prod_{n=1}^{\infty} (1 - q^{4n-2})^2 \right) \\ &= \prod_{n=1}^{\infty} (1 - q^{4n})^2 (1 - q^{4n-2})^4 \\ &= \vartheta_4(2\tau)^2,\end{aligned}$$

as claimed.

APPENDIX C. INTERLUDE: THE THETA SELF-DUALITY POINT $\tau = i$ (I.E. $t = 1$) AND WHAT IT CAN/CANNOT SAY ABOUT RH

C.1. The identity $\vartheta_3 = \vartheta_4 \Rightarrow \lambda = \frac{1}{2}$. We record explicitly the “ $\frac{1}{2}$ ” implication mentioned in the modular-form paper, because it is the cleanest algebraic expression of the self-duality point $\tau = i$.

Proposition C.1 (Self-duality at $\tau = i$ forces $\lambda(i) = 1/2$). *Let $\tau \in \mathbb{H}$ and define $\lambda(\tau) := (\vartheta_2(\tau)/\vartheta_3(\tau))^4$. If $\vartheta_3(\tau) = \vartheta_4(\tau)$ then $\lambda(\tau) = 1/2$. In particular, at $\tau = i$ one has $\vartheta_3(i) = \vartheta_4(i)$ and hence $\lambda(i) = 1/2$.*

Proof. Jacobi’s identity gives

$$\vartheta_3(\tau)^4 = \vartheta_2(\tau)^4 + \vartheta_4(\tau)^4.$$

Assume $\vartheta_3(\tau) = \vartheta_4(\tau)$. Then

$$\vartheta_4(\tau)^4 = \vartheta_2(\tau)^4 + \vartheta_4(\tau)^4 \implies \vartheta_2(\tau)^4 = \vartheta_4(\tau)^4.$$

Therefore

$$\lambda(\tau) = \left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)} \right)^4 = \left(\frac{\vartheta_4(\tau)}{\vartheta_4(\tau)} \right)^4 = \frac{1}{2},$$

since $\vartheta_3(\tau)^4 = \vartheta_2(\tau)^4 + \vartheta_4(\tau)^4 = 2\vartheta_4(\tau)^4$. (Equivalently, $\vartheta_2^4 = \vartheta_4^4$ and $\vartheta_3^4 = 2\vartheta_4^4$.)

Finally, $\tau = i$ is the fixed point of $S : \tau \mapsto -1/\tau$ in the standard fundamental domain, and the theta S -transformations imply $\vartheta_2(i) = \vartheta_4(i)$, hence $\vartheta_3(i)^4 = \vartheta_2(i)^4 + \vartheta_4(i)^4 = 2\vartheta_4(i)^4$, i.e. $\vartheta_3(i) = \vartheta_4(i)$ (on the imaginary axis these values are positive real). $\square$

Remark C.2 (What is special about $\tau = i$). The identity $\lambda(i) = 1/2$ is a *modular symmetry* consequence: $\lambda(-1/\tau) = 1 - \lambda(\tau)$, hence at $\tau = i$ one has $\lambda(i) = 1 - \lambda(i)$. This is structurally analogous to the Ξ -symmetry $\Xi(-z) = \Xi(z)$ and the functional equation $\xi(s) = \xi(1-s)$, but it is not by itself a bridge to RH.

C.2. The Riemann ξ -function is built from a $t \mapsto 1/t$ theta self-duality. The most direct classical connection between zeta and theta is the Mellin transform identity for the Jacobi theta series on the imaginary axis. Define the *Jacobi theta function* on $\mathbb{R}_{>0}$

$$\Theta(t) := \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = \vartheta_3(it), \quad t > 0. \quad (\text{C.1})$$

Then Θ satisfies the modular self-duality (Poisson summation)

$$\Theta(t) = t^{-1/2} \Theta(1/t). \quad (\text{C.2})$$

Note that $t = 1$ is the fixed point of $t \mapsto 1/t$, corresponding to $\tau = i$.

Theorem C.3 (Mellin transform representation of ξ (standard)). *For $(s) > 1$ one has*

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty (\Theta(t) - 1) t^{s/2-1} dt. \quad (\text{C.3})$$

Moreover, by splitting the integral at $t = 1$ and using (C.2), $\xi(s)$ admits the symmetric representation

$$\xi(s) = \frac{1}{2} + \frac{s(s-1)}{2} \int_1^\infty (\Theta(t) - 1) \left(t^{s/2} + t^{(1-s)/2} \right) \frac{dt}{t}, \quad (\text{C.4})$$

which holds for all $s \in \mathbb{C}$ by analytic continuation.

Remark C.4 (Centering at $t = 1$ is forced by symmetry). The symmetric form (C.4) is exactly the analytic reflection of the algebraic fixed point $\tau = i$ in the modular paper: the functional equation $s \leftrightarrow 1-s$ corresponds to the involution $t \leftrightarrow 1/t$ whose fixed point is $t = 1$. In this sense, the “ $\lambda(i) = 1/2$ ” phenomenon is *structurally parallel* to the central role of $t = 1$ in ξ .

C.3. Cosine-transform form of Ξ and why self-duality does not imply RH. Set $\Xi(z) = \xi(\frac{1}{2} + iz)$ as in Definition ???. From (C.4) we obtain an even cosine-transform representation.

Proposition C.5 (Ξ as a cosine transform). *Define for $u \in \mathbb{R}$ the even function*

$$\Phi(u) := e^{u/2} \left(\Theta(e^{2u}) - 1 \right) = e^{u/2} \left(\sum_{n \in \mathbb{Z}} e^{-\pi n^2 e^{2u}} - 1 \right). \quad (\text{C.5})$$

Then $\Phi(u)$ is real, smooth, rapidly decaying as $|u| \rightarrow \infty$, and even: $\Phi(-u) = \Phi(u)$. Moreover,

$$\Xi(z) = \frac{1}{2} + \frac{z^2 + \frac{1}{4}}{2} \int_0^\infty \Phi(u) \cos(zu) du. \quad (\text{C.6})$$

Sketch. Insert $s = \frac{1}{2} + iz$ into (C.4) and change variables $t = e^{2u}$ (so $dt/t = 2 du$). Then

$$t^{s/2} + t^{(1-s)/2} = e^{u/2} (e^{izu} + e^{-izu}) = 2e^{u/2} \cos(zu).$$

The evenness $\Phi(-u) = \Phi(u)$ is a direct rewriting of $\Theta(t) = t^{-1/2} \Theta(1/t)$. This yields (C.6) after collecting constants. $\square$

Remark C.6 (Why the fixed point $u = 0$ (i.e. $t = 1$, $\tau = i$) is not enough). Equation (C.6) shows that RH is equivalent to the assertion that the entire function

$$F(z) := \frac{1}{2} + \frac{z^2 + \frac{1}{4}}{2} \int_0^\infty \Phi(u) \cos(zu) du$$

has only real zeros. The fact that Φ is even (coming from $t \mapsto 1/t$ symmetry and hence from the $t = 1$ fixed point) forces Ξ to be even and real on $\mathbb{R}$, but it does *not* force all zeros to be real.

In other words: the theta self-duality point provides the *symmetry* needed for the functional equation, but RH is a much stronger *real-zero* statement about an entire cosine transform. Proving “cosine transform has only real zeros” typically requires additional deep structure (e.g. membership in the Laguerre–Pólya class, total positivity, or a de Bruijn–Newman type flow argument), not just evenness and decay.

C.4. What would count as genuine RH progress from the $\tau = i$ picture? The modular paper highlights that $t = 1$ (i.e. $\tau = i$) is a universal crossover point for $\lambda(it)$ and a self-dual point for the AGM iteration on theta constants. Here is the cleanest way to translate that into potential RH-relevant analytic progress:

- (a) **Sharper explicit bounds on $\Theta(t) = \vartheta_3(it)$.** From the certificate-architecture viewpoint, any method that yields computable, uniform bounds on $\Theta(t)$ and its derivatives on intervals $t \in [T_1, T_2]$ improves the feasibility of stage certificates for Ξ in rectangles, because Ξ is built from Θ via (C.4) and (C.6).
- (b) **A controlled approximation scheme for $\Phi(u)$.** If one can approximate $\Phi(u)$ by a kernel $\Phi_N(u)$ whose cosine transform is known to have only real zeros, and prove that the approximation preserves real-rootedness (or preserves a de Bruijn–Newman constant bound), that would be genuine structural progress. The modular observations at $\tau = i$ give efficient approximations for theta data, but turning that into a *root-preserving* approximation for the cosine transform is the hard part.
- (c) **A positivity/total-positivity route.** Real-zero theorems for Fourier/cosine transforms often come from total positivity or variation-diminishing properties. The theta self-duality at $u = 0$ guarantees evenness, and the AGM structure provides strong monotonicity and convexity control of theta constants, but no known argument upgrades these facts to the total positivity properties needed for an RH proof.

Remark C.7 (Scope statement). The identity $\vartheta_3 = \vartheta_4 \Rightarrow \lambda = 1/2$ (and the fixed point $t = 1$) explains why $t = 1$ is a privileged symmetry center in theta-based formulas for ξ and Ξ . It does *not* by itself imply RH. Any RH implication would have to pass through a theorem ensuring that the cosine transform (C.6) has only real zeros, which is a substantially stronger analytic property than self-duality.

APPENDIX D. OPTIONAL BRIDGE ADDENDUM: IMPORTING THE MODULAR λ -ODE AND THE $\tau = i$ MIDPOINT INTO THE CERTIFICATE LANGUAGE

This section is optional. It gives a precise way to “patch in” the modular λ -ODE material as an *auxiliary computational oracle* for estimating $\Theta(t) = \vartheta_3(it)$ and hence the theta kernel $\Phi(u)$, without claiming any RH implication.

D.1. Relating $\lambda(it)$, $\vartheta_3(it)$, and $\Theta(t)$. Recall $\Theta(t) = \vartheta_3(it)$. The modular λ -function is

$$\lambda(\tau) = \left(\frac{\vartheta_2(\tau)}{\vartheta_3(\tau)} \right)^4, \quad \tau \in \mathbb{H}.$$

On the imaginary axis, $\tau = it$, $t > 0$, one has $\lambda(it) \in (0, 1)$ and $\lambda(i) = 1/2$. The modular paper proves the ODE

$$\frac{d}{dt}\lambda(it) = -\pi\lambda(it)(1-\lambda(it))\vartheta_3(it)^4 = -\pi\lambda(it)(1-\lambda(it))\Theta(t)^4. \quad (\text{D.1})$$

Proposition D.1 (Solving for $\Theta(t)^4$ in terms of λ). *For $t > 0$,*

$$\Theta(t)^4 = -\frac{1}{\pi} \cdot \frac{\lambda'(it)}{\lambda(it)(1-\lambda(it))}. \quad (\text{D.2})$$

Equivalently, in the logit variable $y(t) = \log(\lambda(it)/(1-\lambda(it)))$,

$$y'(t) = -\pi\Theta(t)^4. \quad (\text{D.3})$$

Remark D.2 (Meaning for certification). Equations (D.2) and the logit identity show that any explicit enclosure scheme for $\lambda(it)$ and $\lambda'(it)$ yields explicit enclosures for $\Theta(t)$. Since ξ and Ξ are built from Θ by (C.4)–(C.6), this gives a principled path to turning modular information into numeric/interval bounds on $\Xi(z)$ on rectangles in the Ξ -plane. That is compatible with the stagewise certificate architecture: it provides a *computable subroutine* for bounding the analytic inputs.

D.2. Midpoint centering at $t = 1$. The midpoint $t = 1$ is singled out by two exact identities:

$$t = 1 \iff \tau = i \iff \lambda(i) = \frac{1}{2} \iff \vartheta_3(i) = \vartheta_4(i).$$

Thus any ODE integration of (D.1) may be conveniently initialized at $t = 1$ with the exact value $\lambda(1) = 1/2$, and propagated to $t > 1$ and $t < 1$ using rigorous interval-ODE methods.

Remark D.3 (No RH consequence without a root-theorem). Even perfect knowledge of $\lambda(it)$ and $\Theta(t)$ does not automatically imply RH; it only improves our ability to *compute or certify* values of $\Xi(z)$ in regions. Converting such computation into RH requires a global argument that all zeros are real, or a certificate system that covers (Definition ??) with verified zero-free windows. The modular midpoint provides a natural computational anchor, not a direct RH proof mechanism.

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